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Distance-regular Subgraphs in a Distance-regular Graph, V

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Let Γ be a distance-regular graph without induced subgraphs $K_{2,1,1}$ and $r = \max\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1)\}$. We give a necessary and sufficient condition for the existence of a strongly closed subgraph which is $(c_{r+1} + a_{r+1})$-regular of diameter $r + 1$ containing a given pair of vertices at distance $r + 1$.

1. INTRODUCTION

Known distance-regular graphs have many subgraphs with high regularity. Sometimes they have distance-regular subgraphs in them. Examples include the Hamming graph, the Johnson graph and the doubled Odd graph. There are strong relations between subgraphs and the original graph. We believe that more research should be done to study the 'global structure' of a distance-regular graph from the information of its 'local structures'. 'Nice' subgraphs would carry a lot of information about the original graph. We have obtained several results on a distance-regular graph from the investigation of its 'local structures'. In fact we have found 'nice' subgraphs and obtained strict restrictions on parameters of the original graph.

In [4, 5, 7], a method of this kind was used and it gave sufficient conditions for the existence of a Moore graph, the collinearity graph of a Moore geometry and the collinearity graph of a generalized polygon in a distance-regular graph as its subgraph, respectively. These examples show the existence of distance-regular subgraphs of the diameter $r + 1$ in a distance-regular graph with $r = r(\Gamma) = \max\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1)\} \geq 2$.

In this paper, we consider general cases and give a necessary and sufficient condition for the existence of a strongly closed subgraph which is $(c_{r+1} + a_{r+1})$-regular of diameter $r + 1$. A strongly closed subgraph of a distance-regular graph is also distance-regular if it is regular. So we can apply the feasibility condition to the subgraph to obtain strong restrictions on the parameter of the original graph.

The reader interested in the 'local-global method' introduced in this paper is advised to read this paper first. This paper covers all the results of [4, 5], even the proofs are shorter and easier.

First we recall our notation and terminologies used in this paper. The reader is referred to [1, 2] for more detailed descriptions of distance-regular graphs.

All graphs considered in this paper are undirected finite graphs without loops or multiple edges. Let Γ be a connected graph with usual distance ∂_Γ. We denote by

$$d_\Gamma := \max\{\partial_\Gamma(x, y) \mid x, y \in \Gamma\}$$

that is called the *diameter* of Γ. Let

$$\Gamma_j(u) := \{x \in \Gamma \mid \partial_\Gamma(u, x) = j\} \quad \text{and} \quad k_\Gamma(u) := |\Gamma_1(u)|.$$

Γ is called a *regular graph of valency* k if $k_\Gamma(u) = k$ for all vertex $u \in \Gamma$.

For two vertices u and x in Γ with $\partial_\Gamma(u, x) = j$, let

$$C(u, x) = C_j(u, x) := \Gamma_{j-1}(u) \cap \Gamma_1(x),$$
$$A(u, x) = A_j(u, x) := \Gamma_j(u) \cap \Gamma_1(x),$$
$$B(u, x) = B_j(u, x) := \Gamma_{j+1}(u) \cap \Gamma_1(x).$$

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Γ is said to be *distance-regular* if

$$c_j(\Gamma) := |C_j(u,x)|, \quad a_j(\Gamma) := |A_j(u,x)| \quad \text{and} \quad b_j(\Gamma) := |B_j(u,x)|$$

depend only on $j = \partial_\Gamma(u,x)$ rather than individual vertices. It is clear that Γ is a regular graph of valency $k_\Gamma = b_0(\Gamma)$ if Γ is distance-regular.

Sometimes we omit the suffix when no confusion occurs. The numbers c_j, a_j and b_j are called the *intersection numbers* of Γ. Let

$$r = r(\Gamma) := \max\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1)\}.$$

Let $w, x, y, z \in \Gamma$ and $\Delta \subseteq \Gamma$. We identify the subset Δ with the induced subgraph on it. Moreover we write

$$\Delta_1(v) = \Gamma_1(v) \cap \Delta \quad \text{and} \quad k_\Delta(v) = |\Delta_1(v)| \qquad \text{for any } v \in \Delta.$$

$K_{2,1,1}$ is a graph on $\{w,x,y,z\}$ with the distance relations $\partial_\Gamma(w,x) = \partial_\Gamma(w,y) = \partial_\Gamma(x,y) = \partial_\Gamma(z,x) = \partial_\Gamma(z,y) = 1$ and $\partial_\Gamma(w,z) = 2$. Let

$$S(x,y) := \{y\} \cup C(x,y) \cup A(x,y),$$
$$P(x,y) := \{v \in \Gamma \mid \partial_\Gamma(x,v) + \partial_\Gamma(v,y) = \partial_\Gamma(x,y)\}$$

that is the set of all vertices on shortest paths connecting x and y, and

$$P(x,\Delta) := \bigcup_{v \in \Delta} P(x,v).$$

For $v \in \Delta$, we say Δ is *strongly closed with respect to* v if $S(v,v') \subseteq \Delta$ for any $v' \in \Delta - \{v\}$. Δ is called *strongly closed* if it is strongly closed with respect to v for all $v \in \Delta$.

A quadruple (w,x,y,z) of vertices is called a *root* if $\partial_\Gamma(w,x) = \partial_\Gamma(y,z) = r+1$, $y \in S(x,w)$ and $z \in S(w,x)$. (See Figure 1.)

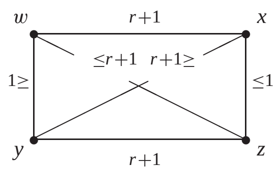

FIGURE 1. A root (w,x,y,z).

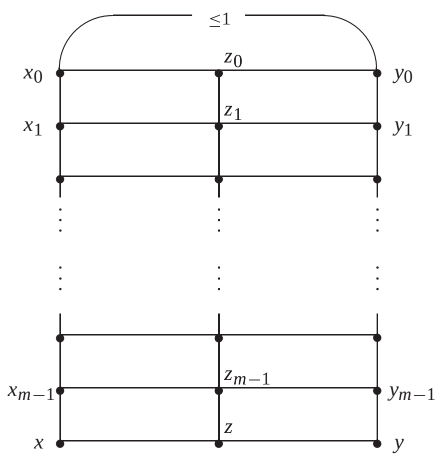

FIGURE 2. A conron (x,y,z) with c-sequences.

In other words, a root (w,x,y,z) is a configuration of vertices at the following distances:

$$\partial_\Gamma(w,x) = \partial_\Gamma(y,z) = r+1, \quad \partial_\Gamma(w,y) \le 1, \quad \partial_\Gamma(x,z) \le 1,$$
$$\partial_\Gamma(w,z) \le r+1 \quad \text{and} \quad \partial_\Gamma(x,y) \le r+1.$$

Note that (x, w, z, y), (y, z, w, x) and (z, y, x, w) are roots if (w, x, y, z) is a root.

A triple (x, y, z) of vertices with $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = r + 1$ is called a *conron* if there exist three sequences of vertices

$$(x_0, x_1, \ldots, x_m = x), \quad (y_0, y_1, \ldots, y_m = y) \quad \text{and} \quad (z_0, z_1, \ldots, z_m = z)$$

such that $\partial_\Gamma(x_0, y_0) \le 1$, $(x_{i-1}, z_{i-1}, x_i, z_i)$ and $(y_{i-1}, z_{i-1}, y_i, z_i)$ are roots for all $1 \le i \le m$. We call these three sequences of vertices *c-sequences*. (See Figure 2.)

In these definitions, the vertices need not be different. For example, (w, x, w, x) is a root and (x, x, w) is a conron if $\partial_\Gamma(w, x) = r + 1$.

The condition (CR) is defined as follows:

$$(CR) : S(x, z) = S(y, z) \text{ for any conron } (x, y, z).$$

In this paper, we prove the following result.

THEOREM 1.1. *Let Γ be a distance-regular graph without induced subgraphs $K_{2,1,1}$ and $r = \max\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1)\}$. Then the following conditions are equivalent:*

(i) (CR) holds.
(ii) Take any u and v in Γ at distance $r + 1$, there exists a strongly closed subgraph $\Delta(u, v)$ which is $(c_{r+1} + a_{r+1})$-regular of diameter $r + 1$ containing u and v.

Before closing this section we roughly sketch the construction of strongly closed subgraphs. Let Γ be a distance-regular graph as in Theorem 1.1. Assume that (CR) holds. Fix a pair of vertices (u, v) in Γ at distance $r + 1$. For any $x, y \in \Gamma_{r+1}(u)$, we define a relation $x \approx y$ if (x, y, u) is a conron. Then this relation $\approx$ becomes an equivalence relation on $\Gamma_{r+1}(u)$. Set Ψ be the equivalence class containing v under this equivalence relation $\approx$. Define $\Delta := P(u, \Psi)$, the subgraph induced on all vertices lying on shortest paths between u and some vertices in Ψ. Then Δ becomes a strongly closed subgraph which is regular of valency $c_{r+1}+a_{r+1}$ of diameter $r + 1$ containing u and v.

We prove our main result in Section 2. In Section 3, we give a sufficient condition for (CR). As an application we show that (CR) holds if $c_{2r+1} = 1$. This gives the second proof of the main results of [4, 5].

2. PROOF OF THE THEOREM

Our purpose in this section is to prove Theorem 1.1. First we introduce the following basic lemma for strongly closed subgraphs.

LEMMA 2.1. *Let $\Omega \subseteq \Gamma$ and $x, y \in \Omega$. Assume that Ω is strongly closed with respect to x. Take any $z \in \Gamma$. If $\partial_\Gamma(x, z) + \partial_\Gamma(z, y) \le \partial_\Gamma(x, y) + 1$, then $z \in \Omega$.*

PROOF. We prove our assertion by induction on $\partial_\Gamma(z, y)$. If $\partial_\Gamma(z, y) \le 1$, then we have $z \in S(x, y) \subseteq \Omega$ from our assumption. Let $\partial_\Gamma(z, y) \ge 2$ and $y' \in C(z, y)$. Since

$$\begin{aligned}\partial_\Gamma(x, y') &\le \partial_\Gamma(x, z) + \partial_\Gamma(z, y') \\ &= \partial_\Gamma(x, z) + \partial_\Gamma(z, y) - 1 \le \partial_\Gamma(x, y),\end{aligned}$$

we have $y' \in S(x, y) \subseteq \Omega$ and $\partial_\Gamma(x, z) + \partial_\Gamma(z, y') \le \partial_\Gamma(x, y) \le \partial_\Gamma(x, y') + \partial_\Gamma(y', y)$. From the inductive assumption, we have $z \in \Omega$. The lemma is proved. □

For the rest of this section, we assume that Γ is a distance-regular graph without the induced subgraphs $K_{2,1,1}$, and $r = r(\Gamma)$. We recall some basic properties of Γ. Note that $b_r > b_{r+1}$ from Proposition 5.4.4 of [2].

LEMMA 2.2. *Let t be an integer with $1 \le t \le r$ and $x, y, z \in \Gamma$ with $\partial_\Gamma(x, z) = t + 1$ and $y \in C(z, x)$. Then the following hold:*

(1) If $t < r$, then $C(y, z) = C(x, z)$, $A(y, z) = A(x, z)$ and $B(y, z) = B(x, z)$.
(2) If there exists $w \in S(x, z) \cap B(y, z)$, then $t = r$ and (x, z, y, w) is a root.
(3) For any $u, v, w \in \Gamma$ with $\partial_\Gamma(u, w) = \partial_\Gamma(v, w) = t$ and $\partial_\Gamma(u, v) = 1$ we have $B(u, w) = B(v, w)$ and $S(u, w) = S(v, w)$.

PROOF. (1) Since $(c_t, a_t, b_t) = (c_{t+1}, a_{t+1}, b_{t+1})$, this is a basic property of distance-regular graphs.

(2) From 1. we have $t = r$. The rest of the assertion is clear.

(3) Suppose $t = 1$. If there exists $w' \in B(u, w) - B(v, w)$, then $\{u, v, w, w'\}$ forms $K_{2,1,1}$ which contradicts our assumption. Thus we have our assertion.

We may assume that $t \ge 2$. Let $(v = v_0, v_1, \ldots, v_t = w)$ be the unique shortest path connecting v and w. Then $u \in A(w, v) = A(v_{t-1}, v) = \cdots = A(v_1, v)$ from (1). This implies $v_1 \in C(w, v) \cap C(w, u)$. Hence $B(u, w) = B(v_1, w) = B(v, w)$ from (1). The desired result follows. $\square$

LEMMA 2.3. *Let $x, z, z' \in \Gamma$ with $\partial_\Gamma(x, z) = r + 1$. If $z' \in S(x, z)$, then there exists $x' \in S(z, x)$ such that (x, z, x', z') is a root.*

PROOF. Since $z' \in S(x, z)$, we have $\partial_\Gamma(x, z') \in \{r, r + 1\}$. If $\partial_\Gamma(x, z') = r + 1$, then we can take x for x'. If $\partial_\Gamma(x, z') = r$, then there exists $x' \in B_r(z', x) - B_{r+1}(z, x) \subseteq S(z, x)$. In each case, (x, z, x', z') is a root. The lemma is proved. $\square$

LEMMA 2.4. *Let (x, y, z, x', y', z') be 6-tuple of vertices such that (x, z, x', z') and (y, z, y', z') are roots. Then (x, y, z) is a conron iff (x', y', z') is a conron.*

PROOF. Suppose (x, y, z) is a conron with c-sequences

$$(x_0, x_1, \ldots, x_m), \quad (y_0, y_1, \ldots, y_m) \quad \text{and} \quad (z_0, z_1, \ldots, z_m).$$

Then we have (x', y', z') is a conron with c-sequences

$$(x_0, x_1, \ldots, x_m, x'), \quad (y_0, y_1, \ldots, y_m, y') \quad \text{and} \quad (z_0, z_1, \ldots, z_m, z')$$

from our assumption. The desired result follows. $\square$

LEMMA 2.5. *Let $u_0 \in \Gamma$ and $(x_0, x_1, \ldots x_t)$ be a path of length t in Γ such that $1 \le t \le r+1$ and $\partial_\Gamma(u_0, x_i) = r + 1 - i$ for all $0 \le i \le t$. Then the following hold:*

(1) There exists a path $(u_0, u_1, \ldots, u_t)$ of length t such that $(u_{i-1}, x_{i-1}, u_i, x_i)$ is a root for all $1 \le i \le t$.
(2) Assume $t \le r$. For any $y_t \in A(u_0, x_t)$, there exist paths $(u_0, u_1, \ldots, u_t)$ and $(y_0, y_1, \ldots, y_t)$ of length t such that (x_0, y_0, u_0) is a conron and $y_t \in P(u_0, y_0)$.

PROOF. (1) Since $b_r > b_{r+1}$, there exists $u_1 \in B_r(x_1, u_0) - B_{r+1}(x_0, u_0)$. Then (u_0, x_0, u_1, x_1) is a root. If $t = 1$, the desired result follows. We prove our assertion by induction on t. Assume $t \ge 2$. Since

$$\begin{aligned} 1 + (r + 1 - j - 1) &= \partial_\Gamma(u_1, u_0) + \partial_\Gamma(u_0, x_{j+1}) \\ &\ge \partial_\Gamma(u_1, x_{j+1}) \\ &\ge \partial_\Gamma(u_1, x_1) - \partial_\Gamma(x_1, x_{j+1}) = (r + 1) - j, \end{aligned}$$

we have $(u_1, x_{j+1}) = r + 1 - j$ for all $0 \le j \le t - 1$. Applying our inductive assumption to u_1 and $(x_1, x_2, \ldots, x_t)$, there exists a path $(u_1, u_2, \ldots, u_t)$ of length $t - 1$ such that $(u_i, x_i, u_{i+1}, x_{i+1})$ is a root for all $1 \le i \le t - 1$. Hence $(u_0, u_1, \ldots, u_t)$ is a path as desired.

(2) Let $(u_0, u_1, \ldots, u_t)$ be a path as in (1). Since $\partial_\Gamma(x_t, u_0) = r+1-t$ and $\partial_\Gamma(x_t, u_t) = r+1$, $u_{i+1} \in B(x_t, u_i)$ for all $0 \le i \le t-1$. Lemma 2.2(3) implies $u_1 \in B(x_t, u_0) = B(y_t, u_0)$. Inductively, we have $u_i \in B(x_t, u_{i-1}) = B(y_t, u_{i-1})$ for all $1 \le i \le t$. Thus $(u_t, u_{t-1}, \ldots, u_0)$ is a path of length t such that $\partial_\Gamma(y_t, u_{t-i}) = r+1-i$ for all $0 \le i \le t$. From (1), there exists a path $(y_t, y_{t-1}, \ldots, y_0)$ of length t such that $(y_i, u_i, y_{i-1}, u_{i-1})$ is a root for any i. Thus (x_0, y_0, u_0) is a conron with c-sequences

$$(x_t, x_{t-1}, \ldots, x_0), \quad (y_t, y_{t-1}, \ldots, y_0) \quad \text{and} \quad (u_t, u_{t-1}, \ldots, u_0).$$

Since

$$r+1 = \partial_\Gamma(u_0, y_0) \le \partial_\Gamma(u_0, y_t) + \partial_\Gamma(y_t, y_0) \le (r+1-t) + t = r+1,$$

we have $y_t \in P(u_0, y_0)$. The lemma is proved. □

For the rest of this section we assume that the condition (CR) holds.

LEMMA 2.6. *Let (x, y, z) be a conron and $y', z' \in \Gamma$ such that (y, z, y', z') is a root. Then there exists $x' \in S(z, x)$ such that (x, z, x', z') is a root and (x', y', z') is a conron.*

PROOF. Since (x, y, z) is a conron and (y, z, y', z') is a root, we have $z' \in S(y, z) = S(x, z)$. From Lemma 2.3, there exists $x' \in S(z, x)$ such that (x, z, x', z') is a root. Then (x', y', z') is a conron from Lemma 2.4. The desired result is proved. □

LEMMA 2.7. *For any vertices $x, y \in \Gamma_{r+1}(z)$, define $x \approx y$ iff (x, y, z) is a conron. Then this is an equivalence relation on $\Gamma_{r+1}(z)$.*

PROOF. Reflexivity and symmetry are clear. We show transitivity. Let (x, y, z) and (y, w, z) be conrons with c-sequences

$$\begin{aligned} &(x_0, x_1, \ldots, x_m), \quad (y_0, y_1, \ldots, y_m), \quad (z_0, z_1, \ldots, z_m) \quad \text{and} \\ &(y'_0, y'_1, \ldots, y'_s), \quad (w'_0, w'_1, \ldots, w'_s), \quad (z'_0, z'_1, \ldots, z'_s), \quad \text{respectively.} \end{aligned}$$

Since (x, y, z) is a conron and $(y, z, y'_{s-1}, z'_{s-1})$ is a root, there exists $x'_{s-1} \in S(z, x)$ such that $(x, z, x'_{s-1}, z'_{s-1})$ is a root and $(x'_{s-1}, y'_{s-1}, z'_{s-1})$ is a conron from Lemma 2.6. Inductively, there exists $x'_{s-i} \in S(z'_{s-i+1}, x'_{s-i+1})$ such that $(x'_{s-i+1}, z'_{s-i+1}, x'_{s-i}, z'_{s-i})$ is a root and $(x'_{s-i}, y'_{s-i}, z'_{s-i})$ is a conron for all $1 \le i \le s$. Then (x, w, z) is a conron with c-sequences

$$\begin{gathered} (x_0, x_1, \ldots, x_m, x'_s, x'_{s-1}, \ldots, x'_0, x'_0, x'_1, \ldots, x'_s = x), \\ (y_0, y_1, \ldots, y_m, y'_s, y'_{s-1}, \ldots, y'_0, w'_0, w'_1, \ldots, w'_s = w), \\ (z_0, z_1, \ldots, z_m, z'_s, z'_{s-1}, \ldots, z'_0, z'_0, z'_1, \ldots, z'_s = z). \end{gathered}$$

The lemma is proved. □

DEFINITION 2.8. *Take any $u, v \in \Gamma$ at distance $r+1$. Let $\approx$ be the equivalence relation on $\Gamma_{r+1}(u)$ as in Lemma 2.7 and $\Psi = \Psi(u, v)$ be the equivalence class containing v. Define*

$$\Delta = \Delta(u, v) := P(u, \Psi),$$

the subgraph induced on the set of all vertices lying on shortest paths between u and some vertices in Ψ.

LEMMA 2.9.

(1) $\Delta(u, v)$ *is strongly closed with respect to* u.
(2) $S(v, u) \subseteq \Delta(u, v)$ *and* $k_\Delta(u) = c_{r+1} + a_{r+1}$.

PROOF. (1) Take any $x \in \Delta - \{u\}$. We show $S(u, x) \subseteq \Delta$. Since $x \in \Delta$, there exists $x' \in \Psi$ such that $x \in P(u, x')$. It is obvious $\{x\} \cup C(u, x) \subseteq P(u, x') \subseteq P(u, \Psi) = \Delta$. We prove $A(u, x) \subseteq \Delta$. If $\partial_\Gamma(u, x) = r + 1$, then (x, x', u) is a conron for any $x' \in A(u, x)$. Hence $x' \in \Psi \subseteq \Delta$ and the desired result follows. So we may assume $\partial_\Gamma(u, x) \leq r$. Let $t := r + 1 - \partial_\Gamma(u, x)$ and $(x' = x_0, x_1, \ldots, x_t = x)$ be a shortest path connecting x' and x. For any $y_t \in A(u, x)$, Lemma 2.5(2) implies that there exist paths $(u = u_0, u_1, \ldots, u_t)$ and $(y_0, y_1, \ldots, y_t)$ of length t such that (x_0, y_0, u_0) is a conron and $y_t \in P(u_0, y_0)$. Since $x_0 = x' \in \Psi$, we have $y_0 \in \Psi$ and $y_t \in P(u, y_0) \subseteq \Delta$.
(2) The first assertion follows from (1) and Lemma 2.1. For any $w \in \Delta_1(u)$, there exists $v' \in \Psi$ such that $w \in P(u, v')$. Then (v', v, u) is a conron and hence $w \in S(v', u) = S(v, u)$. This implies $\Delta_1(u) \cup \{u\} = S(v, u)$. The desired result follows. □

LEMMA 2.10. *If* (x, y, x', y') *is a root, then* $\Delta(x, y) = \Delta(x', y')$.

PROOF. Take any $w \in \Psi(x, y)$. Since (w, y, x) is a conron and (x, y, x', y') is a root, there exists $w' \in S(x, w)$ such that (w, x, w', x') is a root and (w', y', x') is a conron by Lemma 2.6. Then $w' \in \Psi(x', y') \subseteq \Delta(x', y')$. Hence $w \in S(x', w') \subseteq \Delta(x', y')$ from Lemma 2.9(1). This implies $\Psi(x, y) \subseteq \Delta(x', y')$.

Suppose $\Delta(x, y) \not\subseteq \Delta(x', y')$. We take a vertex $z \in \Delta(x, y) - \Delta(x', y')$ that has the maximal distance from x. We set

$$t := \partial_\Gamma(x, z) = \max\{\partial_\Gamma(x, w) \mid w \in \Delta(x, y) - \Delta(x', y')\}.$$

Then we have $z \notin \Psi(x, y)$ since $\Psi(x, y) \subseteq \Delta(x', y')$. As $z \in \Delta(x, y)$, there exists $z' \in \Psi(x, y)$ such that $z \in P(x, z')$. Take $p \in C(z', z) \subseteq B(x, z)$. Then $p \in \Delta(x', y')$ from the maximality of $t = \partial_\Gamma(x, z)$. Since $\Delta(x', y')$ is strongly closed with respect to x' by Lemma 2.9(1), we have $z \in B(x', p)$, for otherwise $z \in S(x', p) \subseteq \Delta(x', y')$ which contradicts our assumption. Since $z \in C(x, p) \cap B(x', p)$, we have $\partial_\Gamma(x, p) = r + 1$, $\partial_\Gamma(x', p) = r$ and (x, p, x', z) is a root from Lemma 2.2. As $p \in \Delta(x, y) \cap \Gamma_{r+1}(x) = \Psi(x, y)$, we have (y, p, x) is a conron and so is (y', z, x') from Lemma 2.4. This implies $z \in \Delta(x', y')$ contradicting our hypothesis. Therefore $\Delta(x, y) \subseteq \Delta(x', y')$. By symmetry, we have $\Delta(x', y') \subseteq \Delta(x, y)$ and the desired result follows. □

PROPOSITION 2.11. Δ *is a strongly closed subgraph of diameter* $r + 1$ *which is regular of valency* $k_\Delta = a_{r+1} + c_{r+1}$. *In particular, it is a distance-regular subgraph. Moreover* $\Delta(x, y) = \Delta$ *for any* $x, y \in \Delta$ *with* $\partial_\Gamma(x, y) = r + 1$.

PROOF. Take any $z \in \Delta - \{u\}$. Let $(u = u_0, u_1, \ldots, u_t = z)$ be a shortest path connecting u and z where $1 \leq t := \partial_\Gamma(u, z) \leq r + 1$. Since $z \in \Delta$, there exists $w \in \Psi$ such that $z \in P(u, w)$. Then we have $\partial_\Gamma(w, u_i) = r + 1 - i$ for all $0 \leq i \leq t$. From Lemma 2.5(1), there exists a path $(w = w_0, w_1, \ldots, w_t)$ of length t such that $(w_{i-1}, u_{i-1}, w_i, u_i)$ is a root for all $1 \leq i \leq t$. Then

$$\Delta = \Delta(u_0, w_0) = \Delta(u_1, w_1) = \cdots = \Delta(u_t, w_t)$$

from Lemma 2.10. Hence $\Delta = \Delta(u_t, w_t)$ is strongly closed with respect to $u_t = z$ and $k_\Delta(z) = c_{r+1} + a_{r+1}$ from Lemma 2.9. This implies that Δ is a strongly closed subgraph of diameter $r + 1$ which is regular of valency $k_\Delta = a_{r+1} + c_{r+1}$. In particular,

$$c_i(\Delta) = c_i(\Gamma), \quad a_i(\Delta) = a_i(\Gamma) \quad \text{and} \quad b_i(\Delta) = k_\Delta - c_i(\Delta) - a_i(\Delta)$$

depend only on i, for all $1 \leq i \leq d_\Delta = r+1$. Therefore the graph Δ is distance-regular.

We prove the second assertion by induction on $h = \partial_\Gamma(u, x)$. The case $u = x$ is clear as $\Psi(x, y) = \Psi$. Assume $h \geq 1$. Let $x' \in C(u, x)$. Since Δ is strongly closed, $x' \in \Delta_1(x) \subseteq S(y, x)$. Then there exists $y' \in S(x, y) \subset \Delta$ such that (x, y, x', y') is a root from Lemma 2.3. Thus

$$\Delta(x, y) = \Delta(x', y') = \Delta$$

from Lemma 2.10 and our inductive hypothesis. The proposition is proved. □

PROOF OF THEOREM 1.1. (i) $\Rightarrow$ (ii): This is a direct consequence of Proposition 2.11.

(ii) $\Rightarrow$ (i): Let (x, y, z) be a conron with c-sequences

$$(x_0, x_1, \ldots, x_m), \quad (y_0, y_1, \ldots, y_m) \quad \text{and} \quad (z_0, z_1, \ldots, z_m).$$

Let $\Delta' := \Delta(x, z)$. Since Δ' is strongly closed, $z_{m-1}, x_{m-1} \in S(x, z) \cup S(z, x) \subseteq \Delta'$. Inductively, we have $z_{m-i}, x_{m-i} \in S(x_{m-i+1}, z_{m-i+1}) \cup S(z_{m-i+1}, x_{m-i+1}) \subseteq \Delta'$ for all $1 \leq i \leq m$. Whence $y_0 \in S(z_0, x_0) \subseteq \Delta'$ and $y_i \in S(z_{i-1}, y_{i-1}) \subseteq \Delta'$ for all $1 \leq i \leq m$. Therefore we obtain $y \in \Delta'$ and

$$S(x, z) = \{z\} \cup \Delta'_1(z) = S(y, z).$$

This completes the proof of Theorem 1.1. □

3. A SUFFICIENT CONDITION FOR (CR)

In this section we give a sufficient condition for (CR) and a class of distance-regular graphs with satisfying (CR).

We start from the definition.

Let Γ be a distance-regular graph without induced subgraphs $K_{2,1,1}$, and $r = r(\Gamma)$.

A *circuit* of length m is a sequence of distinct vertices $(x_0, x_1, \ldots, x_{m-1})$ such that (x_i, x_{i+1}) is an edge of Γ for all $0 \leq i \leq m-1$, where $x_m = x_0$. The *numerical girth* $g = g(\Gamma)$ is the minimal length of circuits no three vertices of which form a triangle. In particular, such a circuit is called a *minimal circuit*. It is well known that

$$g = \begin{cases} 2r+2 & \text{if } c_{r+1} \geq 2, \\ 2r+3 & \text{if } c_{r+1} = 1. \end{cases}$$

LEMMA 3.1. *Assume $d \geq 2r+1$. Let (x, y, z) be a conron and $w \in \Gamma_r(z)$.*

(1) If $g = 2r+3$ and the following conditions (a), (b) *hold, then $\partial_\Gamma(x, w) = 2r+1$ implies $\partial_\Gamma(y, w) = 2r+1$.*

(2) If $g = 2r+2$ and the following conditions (a), (b′) *hold, then $\partial_\Gamma(x, w) = 2r+1$ implies $\partial_\Gamma(y, w) = 2r+1$.*

(a) *Let $\alpha, \beta, \gamma, \delta \in \Gamma$ with $\partial_\Gamma(\alpha, \gamma) = \partial_\Gamma(\beta, \gamma) = r+1$, $\partial_\Gamma(\alpha, \beta) \leq 1$ and $\partial_\Gamma(\gamma, \delta) = r$. Then $\partial_\Gamma(\alpha, \delta) = 2r+1$ implies $\partial_\Gamma(\beta, \delta) = 2r+1$.*

(b) *Let $(\alpha, \beta, \gamma, \delta)$ be a root with $\beta \neq \delta$ and $(\beta = \beta_0, \beta_1, \ldots, \beta_r, \beta_{r+1}, \delta_r, \ldots, \delta_0 = \delta)$ be a minimal circuit. Then $\partial_\Gamma(\alpha, \beta_r) = 2r+1$ implies $\partial_\Gamma(\gamma, \delta_r) = 2r+1$.*

(b′) *Let $(\alpha, \beta, \gamma, \delta)$ be a root with $\beta \neq \delta$ and $(\beta = \beta_0, \beta_1, \ldots, \beta_r, \delta_r, \ldots, \delta_0 = \delta)$ be a minimal circuit. Then $\partial_\Gamma(\alpha, \beta_r) = 2r+1$ implies $\partial_\Gamma(\gamma, \delta_r) = 2r+1$.*

PROOF. Let (x, y, z) be a conron with c-sequences

$$(x_0, x_1, \ldots, x_m = x), \quad (y_0, y_1, \ldots, y_m = y) \quad \text{and} \quad (z_0, z_1, \ldots, z_m = z).$$

If $m = 0$, then the assertion follows from the condition (a). We prove our assertion by induction on the length m of c-sequences of the conrons. Assume $m \geq 1$. Then we have $(x', y', z') = (x_{m-1}, y_{m-1}, z_{m-1})$ is a conron with c-sequences

$$(x_0, x_1, \ldots, x_{m-1}), \ (y_0, y_1, \ldots, y_{m-1}) \quad \text{and} \quad (z_0, z_1, \ldots, z_{m-1}).$$

From the inductive assumption, $\partial_\Gamma(x', w') = 2r + 1$ implies $\partial_\Gamma(y', w') = 2r + 1$ for any $w' \in \Gamma_r(z')$.

Assume $\partial_\Gamma(x, w) = 2r + 1$. We have $\partial_\Gamma(z', w) \in \{r, r + 1\}$ since

$$\begin{aligned} r = (2r + 1) - (r + 1) &\leq \partial_\Gamma(x, w) - \partial_\Gamma(x, z') \\ &\leq \partial_\Gamma(w, z') \\ &\leq \partial_\Gamma(w, z) + \partial_\Gamma(z, z') \leq r + 1. \end{aligned}$$

Case 1. $\partial_\Gamma(z', w) = r$. The above inequality implies $\partial_\Gamma(x, z') = r + 1$. Applying the condition (a) to (w, x, x', z'), we have $\partial_\Gamma(x', w) = 2r + 1$. Then $\partial_\Gamma(y', w) = 2r + 1$ from our inductive assumption. As $r + 1 \geq \partial_\Gamma(y', z) \geq \partial_\Gamma(y', w) - \partial_\Gamma(w, z) = (2r + 1) - r$, we have $\partial_\Gamma(y', z) = r + 1$. Thus $\partial_\Gamma(y, w) = 2r + 1$ by applying the condition (a) to (w, y', y, z).

Case 2. $\partial_\Gamma(z', w) = r + 1$. Let $(z = w_0, \ldots, w_r = w)$ be the unique shortest path connecting z and w. First we treat the case $g = 2r + 3$. Take $v_{r+1} \in A_{r+1}(z', w) - A_r(z, w)$ and let $(z' = v_0, v_1, \ldots, v_{r+1})$ be the unique shortest path connecting z' and v_{r+1}. Then $(w_0, w_1, \ldots, w_r, v_{r+1}, v_r, \ldots, v_0)$ is a minimal circuit. Since (x, z, x', z') is a root with $z \neq z'$ and $\partial_\Gamma(x, w_r) = 2r + 1$, we have $\partial_\Gamma(x', v_r) = 2r + 1$ from the condition (b). Then we have $\partial_\Gamma(y', v_r) = 2r + 1$ from the inductive assumption. Since (y', z', y, z) is a root with $z' \neq z$, we obtain $\partial_\Gamma(y, w_r) = 2r + 1$ from the condition (b). The desired result follows.

For the case $g = 2r + 2$, take $v_r \in C_{r+1}(z', w) - C_r(z, w)$ and let $(z' = v_0, v_1, \ldots, v_r)$ be the unique shortest path connecting z' and v_r. Then $(w_0, w_1, \ldots, w_r, v_r, \ldots, v_0)$ is a minimal circuit. The rest of the proof is the same as the previous one using the condition (b′) instead of (b). □

PROPOSITION 3.2. *Assume* $d \geq 2r + 1$.

1. *If* $g = 2r + 3$ *and the conditions* (a), (b) *of Lemma 3.1 hold, then* (CR) *holds.*
2. *If* $g = 2r + 2$ *and the conditions* (a), (b′) *of Lemma 3.1 hold, then* (CR) *holds.*

PROOF. Let (x, y, z) be a conron. Take any $w_1 \in B(x, z)$. Let $w_i \in B(x, w_{i-1})$ for all $2 \leq i \leq r$. Then $\partial_\Gamma(z, w_r) = r$ and $\partial_\Gamma(x, w_r) = 2r + 1$. From Lemma 3.1 we have $\partial_\Gamma(y, w_r) = 2r + 1$. As

$$\begin{aligned} (r + 1) + 1 &\geq \partial_\Gamma(y, z) + \partial_\Gamma(z, w_1) \\ &\geq \partial_\Gamma(y, w_1) \\ &\geq \partial_\Gamma(y, w_r) - \partial_\Gamma(w_r, w_1) \geq (2r + 1) - (r - 1), \end{aligned}$$

we obtain $\partial_\Gamma(y, w_1) = r+2$ and $w_1 \in B(y, z)$. This implies $B(x, z) \subseteq B(y, z)$. Comparing the sizes of both sides, we have $B(x, z) = B(y, z)$ and hence $S(x, z) = S(y, z)$. The proposition is proved. □

To show the efficiently of Proposition 3.2, in the following we reprove main results of [4, 5]. They are restated as follows: If $c_{2r+1} = 1$, then (CR) holds.

Let Γ be a distance-regular graph with $c_{2r+1} = 1$. First we recall several basic properties.

LEMMA 3.3. *Let* s, t *be integers with* $t \leq 2r$ *and* $s \leq 2r + 1 - t$. *Let* $x, y, z, w \in \Gamma$ *with* $\partial_\Gamma(x, y) \leq 1$ *and* $\partial_\Gamma(x, z) = \partial_\Gamma(y, z) = t$. *Then the following hold:*

(1) $B(x, z) = B(y, z)$ *and* $C(x, z) \cup A(x, z) = C(y, z) \cup A(y, z)$.
(2) If $\partial_\Gamma(z, w) = s$ *and* $\partial_\Gamma(x, w) = t + s$, *then* $\partial_\Gamma(y, w) = t + s$.

PROOF. (1) Suppose there exists $x' \in B(x, z) - B(y, z)$. Then $x' \in \Gamma_{t+1}(x) \cap \Gamma_t(y)$ and $z \in C(x, x') - C(y, x')$ which contradicts $c_t = c_{t+1}$. Thus we have $B(x, w) \subseteq B(y, w)$. Comparing the sizes of both sides, the assertion follows.

(2) Let $(z = z_0, z_1, \ldots, z_s = w)$ be the unique shortest path connecting z and w. Then $\partial_\Gamma(x, z_i) = t + i$ for all $0 \le i \le s$ from our hypothesis. Hence $z_1 \in B(x, z_0) = B(y, z_0)$ and $\partial_\Gamma(x, z_1) = \partial_\Gamma(y, z_1) = t + 1$ from (1). Inductively, we have

$$z_i \in B(x, z_{i-1}) = B(y, z_{i-1}) \quad \text{and} \quad \partial_\Gamma(x, z_i) = \partial_\Gamma(y, z_i) = t + i$$

for all $1 \le i \le s$. The desired result is proved. □

LEMMA 3.4. *Let* $w \in \Gamma$ *and* $(x_0, x_1, \ldots, x_r, y_{r+1}, y_r, \ldots, y_1, y_0)$ *be a minimal circuit with* $\partial_\Gamma(w, x_0) = \partial_\Gamma(w, y_0) = r + 1$. *If* $\partial_\Gamma(w, x_r) = 2r + 1$, *then* $\partial_\Gamma(w, y_r) = 2r + 1$.

PROOF. Let $(w = w_0, w_1, \ldots, w_{r+1} = x_0)$ be the unique shortest path connecting w and x_0. Then $\partial_\Gamma(x_r, w_i) = 2r + 1 - i$ for all $0 \le i \le r + 1$. From Lemma 3.3 (1), we have $\{w_1\} = C(x_0, w) \subseteq C(y_0, w) \cup A(y_0, w)$. Thus $\partial_\Gamma(y_0, w_1) \in \{r, r + 1\}$.

Case 1. $\partial_\Gamma(y_0, w_1) = r + 1$. Since $C(w, x_r) = \{x_{r-1}\}$, we have $\partial_\Gamma(w, y_{r+1}) \in \{2r + 1, 2r + 2\}$. Then there exists an integer q with $0 \le q \le r + 1$ such that $\partial_\Gamma(w, y_i) = r + 1 + i$ for all $0 \le i \le q$ and $\partial_\Gamma(w, y_i) = r + i$ for all $q < i \le r + 1$. It is enough to show that $r \le q$. Assume $q \le r - 1$ to derive a contradiction. Note that $\partial_\Gamma(w_1, x_r) = 2r$.

If $q = 0$, then we have $y_1 \in A(w, y_0) - \{x_0\} \subseteq A(w_1, y_0)$ from Lemma 3.3(1). This implies $\partial_\Gamma(w_1, y_1) = \partial_\Gamma(w, y_1) = r + 1$.

If $q \ge 1$, then we have $\partial_\Gamma(w_1, y_{q-1}) = \partial_\Gamma(w, y_{q-1}) = r + q$ and $\partial_\Gamma(w_1, y_q) = \partial_\Gamma(w, y_q) = r + q + 1$ from Lemma 3.3(2). Since $\{y_{q-1}\} = C(w, y_q) = C(w_1, y_q)$ and $B(w, y_q) = B(w_1, y_q)$ from Lemma 3.3(1), we have $y_{q+1} \in A(w, y_q) = A(w_1, y_q)$.

In each case we have $\partial_\Gamma(w, y_{q+1}) = \partial_\Gamma(w_1, y_{q+1}) = r + q + 1$. Hence Lemma 3.3(2) implies $\partial_\Gamma(w_1, y_r) = 2r$ and $\partial_\Gamma(w_1, y_{r+1}) = 2r + 1$. We have $\{x_r, y_r\} \subseteq C(w_1, y_{r+1})$ which contradicts $c_{2r+1} = 1$.

Case 2. $\partial_\Gamma(y_0, w_1) = r$. From Lemma 2.2(1) we have

$$y_0 \in A(w_1, x_0) = A(w_2, x_0) = \cdots = A(w_r, x_0)$$

and $\partial_\Gamma(y_0, w_i) = r + 1 - i$ for all $0 \le i \le r$. Since $\partial_\Gamma(w_r, x_0) = \partial_\Gamma(x_0, y_0) = \partial_\Gamma(w_r, y_0) = 1$ and $\partial_\Gamma(x_0, y_r) = r + 1$, Lemma 3.3(2) shows that $\partial_\Gamma(w_r, y_r) = r + 1$. Then we have $\partial_\Gamma(w_r, y_{r+1}) = r + 1$ for otherwise $\partial_\Gamma(w_r, y_{r+1}) = r + 2$ and $\{y_r, x_r\} \subseteq C_{r+2}(w_r, y_{r+1})$ contradicting $c_{r+2} = 1$. Since $\partial_\Gamma(x_r, w_r) = \partial_\Gamma(y_{r+1}, w_r) = r + 1$ and $\partial_\Gamma(x_r, w) = 2r + 1$, we have $\partial_\Gamma(y_{r+1}, w) = 2r + 1$ from Lemma 3.3(2). Similarly $\partial_\Gamma(y_{r+1}, w_r) = \partial_\Gamma(y_r, w_r) = r + 1$ and $\partial_\Gamma(y_{r+1}, w) = 2r + 1$ implies $\partial_\Gamma(y_r, w) = 2r + 1$ from Lemma 3.3(2). This is the desired result. □

PROPOSITION 3.5. *If* $c_{2r+1} = 1$, *then* (CR) *holds.*

PROOF. Lemma 3.3(2) shows that condition (a) of Lemma 3.1 holds. We prove that condition (b) of Lemma 3.1 also holds.

Let $(\alpha, \beta, \gamma, \delta)$ be a root with $\beta \ne \delta$ and $(\beta = \beta_0, \beta_1, \ldots, \beta_r, \delta_{r+1}, \delta_r, \ldots, \delta_0 = \delta)$ be a minimal circuit. Assume that $\partial_\Gamma(\alpha, \beta_r) = 2r + 1$. Since $\delta \in S(\alpha, \beta)$, $\partial_\Gamma(\alpha, \delta) \in \{r, r + 1\}$.

If $\partial_\Gamma(\alpha, \delta) = r + 1$, then Lemma 3.4 shows $\partial_\Gamma(\alpha, \delta_r) = 2r + 1$. Then $\partial_\Gamma(\gamma, \delta_r) = 2r + 1$ from Lemma 3.3(2).

If $\partial_\Gamma(\alpha, \delta) = r$, then $\gamma \notin C(\delta, \alpha) = C(\beta, \alpha)$. Hence $\partial_\Gamma(\gamma, \beta) = r + 1$. From Lemma 3.3(2), we have $\partial_\Gamma(\gamma, \beta_r) = 2r + 1$. Then $\partial_\Gamma(\gamma, \delta_r) = 2r + 1$ from Lemma 3.4.

Hence the condition (b) of Lemma 3.1 holds. The desired result is a direct consequence of Proposition 3.2(1). □

REMARKS.

1. From a direct consequence of Theorem 1.1 and Proposition 3.5, there exists the collinearity graph of a Moore geometry (in particular it is a Moore graph if $a_1 = 0$) as a subgraph in a distance-regular graph with $c_{2r+1} = 1$. This result is proved by Brouwer and Ivanov for the case $r = 1$ [2, Proposition 4.3.11] and by the author for the general cases [4–6].
2. The main result of [7] shows that a distance-regular graph with $r(\Gamma) = r \geq 2, a_1 > 0$ and $a_i = c_i a_1$ for all $i \leq 2r$ has the collinearity graph of a generalized $2(r+1)$-gon of order $(a_1 + 1, c_{r+1} - 1)$ as a subgraph. This result can be proved by the method introduced in this paper. One has only to show that the conditions (a) and (b′) of Lemma 3.1 hold.
3. The method used in this paper is very powerful and useful for constructing distance-regular subgraphs in a distance-regular graph. We strongly believe that a lot of good results will be obtained by this method. In fact, we can show the existence of distance-regular subgraphs in a distance-regular graph if the conditions (a), (b) or (a), (b′) of Lemma 3.1 hold. The author has succeeded in showing that an interesting class of distance-regular graphs satisfies the condition (CR). This result will be introduced in the forthcoming paper [8].

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